Sample Final Exam Problems, MA3025, Fall 2004

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Each of the following problems has appeared at least once as a final exam problem in MA3025. If you select between five and seven of these at random, there is a high probability that you'll be looking at a reasonable practice exam. Please do not lose sight of the fact that knowing how to solve these problems does not guarantee that you will get a perfect score on the final!

1. Let $A = \{1, 2, 3, 4\}$. For each of the following collections of properties, construct a relation on $A$ that possesses each of the properties in that collection.

   (a) Reflexivity, Symmetry, Antisymmetry, Transitivity.

   (b) Reflexivity, Antisymmetry, Transitivity.

   (c) Reflexivity, Symmetry, Transitivity.

2. Let $A \neq \emptyset$, and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be symmetric relations on $A$. Claim: if $\mathcal{R}_1 \circ \mathcal{R}_2 \subseteq \mathcal{R}_2 \circ \mathcal{R}_1$, then $\mathcal{R}_1 \circ \mathcal{R}_2 = \mathcal{R}_2 \circ \mathcal{R}_1$. Is this claim true? Either prove that it is true, or provide a counterexample.

3. Let $f : A \to B$, where $A$ and $B$ are finite sets. Prove the following statements.

   (a) If $f$ is a bijection, then $|A| = |B|$.

   (b) If $|A| = |B|$, then $f$ is one-to-one if and only if $f$ is onto.

4. Let $f : A \to B$, and $g : C \to D$. Define $h : A \times C \to B \times D$ by $h(a, c) = (f(a), g(c))$. Prove that $h$ is bijective iff both $f$ and $g$ are bijective.

5. Let $A \subset \mathbf{Z}^+$, and define the relation $\preceq$ on $A$ by $a \preceq b$ iff $a|b$. Show that $\preceq$ is a partial order relation.

6. Let $\mathcal{R}$ be the relation on $\mathbf{R} - \mathbf{Z}$ defined by $x\mathcal{R}y$ if and only if $\lfloor x \rfloor = \lceil y \rceil$. Classify $\mathcal{R}$ in terms of reflexivity, symmetry, antisymmetry, and transitivity.

7. How many positive integers between 1 and 36 (inclusive) must one choose to guarantee that we have a pair that are not relatively prime?

8. Let $n \in \mathbf{Z}^+$. Find all real numbers $x$ such that $n\lfloor x \rfloor = \lfloor nx \rfloor$.

9. It is straightforward to show that if we choose any three positive integers, we have two whose sum is even. (This is really the sock problem.) Similarly if we choose any five ordered pairs of positive integers, we have two, say $(x_1, x_2), (y_1, y_2)$, with the property that both $x_1 + x_2$ and $y_1 + y_2$ are even. The general case appears to be the following: given any $2^k + 1$ $k$-tuples of positive integers, we are guaranteed a pair of $k$-tuples $(x_1, \ldots, x_k), (y_1, \ldots, y_k)$ such that $x_i + y_i$ is even for all $1 \leq i \leq k$. Prove this result.

10. Is $p \leftrightarrow (q \leftrightarrow (p \leftrightarrow q))$ a tautology? Explain.

11. Define a relation $R$ on $\mathbf{R}$ by $xRy$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.

    (a) Classify $R$ in terms of reflexivity, symmetry, antisymmetry, and transitivity.

    (b) Is $R$ an equivalence relation? If so, what are the equivalence classes?

12. The *Lucas numbers* $L_n$ are defined by $L_0 = 2,\ L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Prove that $\displaystyle\sum_{k=0}^{n} L_k = L_{n+2} - 1$ for all $n \in \mathbf{N}$.

13. Let $\mathcal{R}$ be the relation on $\mathbf{Z} \times \mathbf{Z}$ defined by $(a, b)\mathcal{R}(c, d)$ iff $a \leq c$ and $b \geq d$.

(a) Prove that $\mathcal{R}$ is a partial order on $\mathbf{Z} \times \mathbf{Z}$.

(b) Let $\mathbf{Z}_n = \{0, 1, \ldots, n-1\}$, and let $\mathcal{R}^{(n)}$ be the *restriction* of $\mathcal{R}$ to $\mathbf{Z}_n \times \mathbf{Z}_n$, i.e., $\mathcal{R}^{(n)} = \{((a,b),(c,d)) | a,b,c,d \in \mathbf{Z}_n \wedge ((a,b),(c,d)) \in \mathcal{R}\}$. Draw the Hasse diagram for $\mathcal{R}^{(3)}$.

(c) Use topological sort to construct a linear order compatible with $\mathcal{R}^{(3)}$.

(d) Is $\mathcal{R}$ a lattice? If so, given pairs $(a,b),(c,d) \in \mathbf{Z} \times \mathbf{Z}$, what are $\text{lub}((a,b),(c,d))$ and $\text{glb}((a,b),(c,d))$? If $\mathcal{R}$ is not a lattice, why not?

14. A bakery has plain croissants, cherry croissants, almond croissants, and apple croissants. How many ways are there to choose

    (a) A dozen croissants?

    (b) A dozen croissants containing at least three almond croissants?

    (c) A dozen croissants containing at most three apple croissants?

    (d) A dozen croissants containing at least three almond croissants and at most three apple croissants?

15. Classify each of the following relations on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$, in terms of reflexivity, symmetry, antisymmetry, and transitivity.

    (a) $\{(f,g) | f(1) = g(1)\}$.

    (b) $\{(f,g) | (f(0) = g(0)) \vee (f(1) = g(1))\}$

    (c) $\{(f,g) | (f(0) = g(1)) \wedge (f(1) = g(0))\}$

16. Let $n \in \mathbf{Z}$.

    (a) Show that $n = 3j + 7k$ for some $j, k \in \mathbf{Z}$.

    (b) Use induction to show that if $n \geq 12$, then $n = 3j + 7k$ for $j, k \in \mathbf{N}$.

17. Let $A = \{1, 2, 3\}$. Define $\preceq$ on $A \times A$ by $(a_1, a_2) \preceq (b_1, b_2)$ iff $a_1 \geq b_1$ and $a_2 | b_2$.

    (a) Show that $(A \times A, \preceq)$ is a poset.

    (b) Sketch the Hasse diagram for $(A \times A, \preceq)$.

    (c) Use topological sort to find a linear ordering of $A \times A$ that is compatible with $\preceq$.

    (d) Is $(A \times A, \preceq)$ a lattice? Why, or why not?

18. Use a combinatorial proof to prove that, for all $k \leq r \leq n$ in $\mathbf{N}$,

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$